JEE ADVANCED

Single Correct Answer Type

1. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then

a. $m^2 - n^2 = 4mn$ **b.** $m^2 + n^2 = 4mn$

c. $m^2 - n^2 = m^2 + n^2$ **d.** $m^2 - n^2 = 4\sqrt{mn}$

(IIT-JEE 1970)

2. If $\alpha + \beta + \gamma = 2\pi$, then

a. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

b. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

c. $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

d. none of these **(IIT-JEE 1979)**

3. If $\tan\theta=-\frac{4}{3}$, then $\sin\theta$ is

a. $-\frac{4}{5}$ but not $\frac{4}{5}$ b. $-\frac{4}{5}$ or $\frac{4}{5}$

c. $\frac{4}{5}$ but not $-\frac{4}{5}$ d. none of these

(IIT-JEE 1979)

4. Given $A=\sin^2\theta+\cos^4\theta$, then for all real θ,

a. $1\le A\le 2$ b. $3/4\le A\le 1$

c. $13/16\le A\le 1$ d. $3/4\le A\le 13/16$

(IIT-JEE 1980)

5. The value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is

a. 1/4 b. 3/4 c. 1/8 d. 3/8

(IIT-JEE 1984)

6. The value of the expression $\sqrt{3}$ cosec 20° – sec 20° is equal to

a. 2 b. 2 sin 20°/sin 40°

c. 4 d. 4 sin 20°/sin 40°

(IIT-JEE 1988)

7. Let $0<x<\pi/4$, then $(\sec 2x-\tan 2x)$ equals

a. $\tan\left(x-\frac{\pi}{4}\right)$ b. $\tan\left(\frac{\pi}{4}-x\right)$

c. $\tan\left(x+\frac{\pi}{4}\right)$ d. $\tan^2\left(x+\frac{\pi}{4}\right)$

(IIT-JEE 1994)

8. Let n be a positive integer such that $\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2}$. Then

a. $6\le n\le 8$ b. $4<n\le 8$

c. $4\le n\le 8$ d. $4<n<8$ **(IIT-JEE 1994)**

9. $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$ is equal to

a. 11 b. 12 c. 13 d. 14

(IIT-JEE 1995)

10. $\sec^2\theta=\frac{4xy}{(x+y)^2}$ is true if and only if

a. $x+y\ne 0$ b. $x=y, x\ne 0$

c. $x=y$ d. $x\ne 0, y\ne 0$

(IIT-JEE 1996)

11. Let $f(\theta)=\sin\theta(\sin\theta+\sin 3\theta)$. Then $f(\theta)$ is

a. ≥ 0 only when $\theta\ge 0$ b. ≤ 0 for all real θ

c. ≥ 0 for all real θ d. ≤ 0 only when $\theta\le 0$

(IIT-JEE 2000)

12. If $\alpha+\beta=\pi/2$ and $\beta+\gamma=\alpha$, then $\tan\alpha$ equals

a. $2(\tan\beta+\tan\gamma)$ b. $\tan\beta+\tan\gamma$

c. $\tan\beta+2\tan\gamma$ d. $2\tan\beta+\tan\gamma$

(IIT-JEE 2001)

13. The maximum value of $(\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n)$ under the restrictions $0\le\alpha_1,\alpha_2,\ldots,\alpha_n\le\pi/2$ and $(\cot\alpha_1)(\cot\alpha_2)\cdots(\cot\alpha_n)=1$ is

a. $1/2^{n/2}$ b. $1/2^n$ c. $1/2n$ d. 1

(IIT-JEE 2001)

14. Given both θ and ϕ are acute angles and $\sin\theta=1/2$, $\cos\phi=1/3$, then the value of $\theta+\phi$ belongs to

a. $\left(\frac{\pi}{3},\frac{\pi}{2}\right]$ b. $\left(\frac{\pi}{2},\frac{2\pi}{3}\right]$

c. $\left(\frac{2\pi}{3},\frac{5\pi}{6}\right]$ d. $\left(\frac{5\pi}{6},\pi\right]$

(IIT-JEE 2004)

15. Let $\theta\in(0,\pi/4)$ and $t_1=(\tan\theta)^{\tan\theta}$, $t_2=(\tan\theta)^{\cot\theta}$, $t_3=(\cot\theta)^{\tan\theta}$ and $t_4=(\cot\theta)^{\cot\theta}$, then

a. $t_1>t_2>t_3>t_4$ b. $t_4>t_3>t_1>t_2$

c. $t_3>t_1>t_2>t_4$ d. $t_2>t_3>t_1>t_2$

(IIT-JEE 2006)

16. Let $P=\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\}$ and $Q=\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\}$ be two sets. Then

a. $P\subset Q$ and $Q-P\ne\varnothing$ b. $Q\not\subset P$

c. $P\not\subset Q$ d. $P=Q$ **(IIT-JEE 2011)**

Multiple Correct Answers Type

1. The expression $3\left[\sin^4\left(\frac{3}{2}\pi-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{1}{2}\pi+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to

a. 0 b. 1 c. 3 d. none of these

(IIT-JEE 1984)

2. For $0<\phi\le\pi/2$, if $x=\sum_{n=0}^{\infty}\cos^{2n}\phi$, $y=\sum_{n=0}^{\infty}\sin^{2n}\phi$, $z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$, then

a. $xyz=xz+y$ b. $xyz=xy+z$

c. $xyz=x+y+z$ d. $xyz=yz+x$

(IIT-JEE 1992)

3. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where α,β,γ are real numbers satisfying $\alpha+\beta+\gamma=\pi$ is

a. positive b. zero

c. negative d. –3 **(IIT-JEE 1995)**

4. Which of the following number(s) is/are rational?

a. sin 15° b. cos 15°

c. sin 15° cos 15° d. sin 15° cos 75°

(IIT-JEE 1998)

5. For a positive integer n, let $f_n(\theta) = (\tan\theta/2)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\cdots(1+\sec 2^n\theta)$. Then

a. $f_2(\pi/16) = 1$ b. $f_3(\pi/32) = 1$
c. $f_4(\pi/64) = 1$ d. $f_5(\pi/128) = 1$

(IIT-JEE 1999)

6. If $\dfrac{\sin^4 x}{2} + \dfrac{\cos^4 x}{3} = \dfrac{1}{5}$, then

a. $\tan^2 x = \dfrac{2}{3}$ b. $\dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{1}{125}$

c. $\tan^2 x = \dfrac{1}{3}$ d. $\dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{2}{125}$

(IIT-JEE 2009)

7. Let $f: (-1, 1) \to R$ be such that $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then the value(s) of $f\left(\dfrac{1}{3}\right)$ is (are)

a. $1-\sqrt{\dfrac{3}{2}}$ b. $1+\sqrt{\dfrac{3}{2}}$ c. $1-\sqrt{\dfrac{2}{3}}$ d. $1+\sqrt{\dfrac{2}{3}}$

(IIT-JEE 2012)

Matching Column Type

1. Match List I with List II and select the correct answer using the code given below the lists :

List I	List II
(p) $\left(\dfrac{1}{y^2}\left(\dfrac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)^2 + y^4\right)^{1/2}$ takes value	(1) $\dfrac{1}{2}\sqrt{\dfrac{5}{3}}$
(q) If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible value of $\cos\dfrac{x-y}{2}$ is	(2) $\sqrt{2}$
(r) If $\cos\left(\dfrac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\dfrac{\pi}{4} + x\right)\cos 2x$ then possible value of $\sec x$ is	(3) 1/2
(s) If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin(\tan^{-1}(x\sqrt{6}))$, $x \neq 0$, then possible value of x is	(4) 1

Codes:

	(p)	(q)	(r)	(s)
a.	(4)	(3)	(1)	(2)
b.	(4)	(3)	(2)	(1)
c.	(3)	(4)	(2)	(1)
d.	(3)	(4)	(1)	(2)

(IIT-JEE 2013)

Integer Answer Type

1. The positive integer value of $n > 3$ satisfying the equation $\dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\dfrac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\dfrac{3\pi}{n}\right)}$ is (IIT-JEE 2011)

2. The maximum value of the expression $\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is (IIT-JEE 2010)

Fill in the Blanks Type

1. Suppose $\sin^3 x \sin 3x = \sum_{m=0}^{n} C_m \cos mx$ is an identity in x, where $C_0, C_1, \ldots, C_n$ are constants, and $C_n \neq 0$, then the value of n is _____. (IIT-JEE 1981)

2. The side of a triangle inscribed in a given circle subtends angles α, β, and γ at the center. The minimum value of the arithmetic mean of $\cos\left(\alpha+\dfrac{\pi}{2}\right)$, $\cos\left(\beta+\dfrac{\pi}{2}\right)$, and $\cos\left(\gamma+\dfrac{\pi}{2}\right)$ is equal to_____. (IIT-JEE 1987)

3. The value of
$$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$$
is equal to _____. (IIT-JEE 1991)

4. If $K = \sin(\pi/18)\sin(5\pi/18)\sin(7\pi/18)$, then the numerical value of K is _____. (IIT-JEE 1993)

5. If $A > 0$, $B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is _____. (IIT-JEE 1993)

6. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in H.P., then $\cos x \sec\left(\dfrac{y}{2}\right) =$ _____. (IIT-JEE 1997)

True/False Type

1. If $\tan A = \dfrac{1-\cos B}{\sin B}$, then $\tan 2A = \tan B$.

(IIT-JEE 1983)

Subjective Type

1. If $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, find the possible values of $(\alpha+\beta)$. (IIT-JEE 1978)

2. a. Draw the graph of $y = \dfrac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\dfrac{\pi}{2}$ to $x = \dfrac{\pi}{2}$. (IIT-JEE 1979)

b. If $\cos(\alpha+\beta) = \dfrac{4}{5}$, $\sin(\alpha-\beta) = \dfrac{5}{13}$, and α, β lie between 0 and $\pi/4$, find $\tan 2\alpha$.

3. Prove that $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ lies between -4 and 10. **(IIT-JEE 1979)**

4. Given $\alpha + \beta - \gamma = \pi$, prove that $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$. **(IIT-JEE 1980)**

5. For all θ in $[0, \pi/2]$ show that $\cos(\sin\theta) \geq \sin(\cos\theta)$. **(IIT-JEE 1981)**

6. Without using tables, prove that $(\sin 12°)(\sin 48°)(\sin 54°) = 1/8$. **(IIT-JEE 1980)**

7. Show that $16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right) = 1$. **(IIT-JEE 1983)**

8. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$. **(IIT-JEE 1988)**

9. ABC is a triangle such that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = 1/2$. If A, B and C are in A.P. determine the values of A, B, and C. **(IIT-JEE 1990)**

10. Show that the value of $\frac{\tan x}{\tan 3x}$, wherever defined, never lies between $\frac{1}{3}$ and 3. **(IIT-JEE 1992)**

11. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n} = -\frac{n}{2}$, where $n \geq 3$ is an integer. **(IIT-JEE 1997)**

12. Find the range of values of t for which

$$2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$ **(IIT-JEE 2005)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. d. 2. a. 3. b. 4. b. 5. c.
6. c. 7. b. 8. d. 9. c. 10. b.
11. c. 12. c. 13. a. 14. b. 15. b.
16. d.

Multiple Correct Answers Type

1. b. 2. b., c. 3. c. 4. c.
5. a., b., c., d. 6. a., b. 7. a., b.

Matching Column Type

1. b.

Integer Answer Type

1. 7 2. 2

Fill in the Blanks Type

1. $n = 6$ 2. $-\frac{\sqrt{3}}{2}$ 3. 1/64 4. 1/8 5. 1/3
6. $\pm\sqrt{2}$

True/False Type

1. True

Subjective Type

1. $n\pi + \frac{\pi}{4}, n \in Z$

2. b. 56/33
9. $A = 45°, B = 60°, C = 75°$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. d. From the given relations,

$$m + n = 2\tan\theta,\ m - n = 2\sin\theta.$$

Thus, $m^2 - n^2 = 4\tan\theta\sin\theta$ (i)

Also $\sqrt{mn} = 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sin\theta\tan\theta$ (ii)

From Eqs. (i) and (ii), we get $m^2 - n^2 = 4\sqrt{mn}$.

2. a. $\alpha + \beta + \gamma = 2\pi$ or $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi$

$\therefore \quad \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(\pi - \frac{\gamma}{2}\right) = -\tan\frac{\gamma}{2}$

$\Rightarrow \quad \frac{\tan\alpha/2 + \tan\beta/2}{1 - \tan\alpha/2\tan\beta/2} = -\tan\gamma/2$

$\Rightarrow \quad \tan\alpha/2 + \tan\beta/2 + \tan\gamma/2$
$= \tan\alpha/2\ \tan\beta/2\ \tan\gamma/2$

3. b. $\tan\theta = \frac{-4}{3}$

$\Rightarrow \quad \theta \in$ 2nd quadrant or 4th quadrant

$\Rightarrow \quad \sin\theta = \pm 4/5$

If $\theta \in$ 2nd quadrant, $\sin\theta = 4/5$

If $\theta \in$ 4th quadrant, $\sin\theta = -4/5$

4. b. We have

$$A = \sin^2\theta + \cos^4\theta$$
$$= 1 - \cos^2\theta + \cos^4\theta$$
$$= 1 + (\cos^4\theta - \cos^2\theta)$$
$$= 1 + \left(\cos^2\theta - \frac{1}{2}\right)^2 - \frac{1}{4}$$
$$= \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2 \geq \frac{3}{4}$$

Now $0 \leq \cos^2\theta \leq 1$

$\Rightarrow \quad -\frac{1}{2} \leq \cos^2\theta - \frac{1}{2} \leq \frac{1}{2}$

$\Rightarrow \quad 0 \leq \left(\cos^2\theta - \frac{1}{2}\right)^2 \leq \frac{1}{4}$

$\Rightarrow \quad \frac{3}{4} \leq \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \leq 1$

$\Rightarrow \quad \frac{3}{4} \leq A \leq 1$

Alternative Method:

We have $A = \sin^2\theta + \cos^4\theta$

$$= 1 - \cos^2\theta + \cos^4\theta$$
$$= 1 - \cos^2\theta(1 - \cos^2\theta)$$
$$= 1 - \cos^2\theta\sin^2\theta$$
$$= 1 - \frac{1}{4}\sin^2 2\theta$$

Now $0 \leq \frac{1}{4}\sin^2 2\theta \leq \frac{1}{4}$

$\Rightarrow \quad -\frac{1}{4} \leq -\frac{1}{4}\sin^2 2\theta \leq 0$

$\Rightarrow \quad \frac{3}{4} \leq 1 - \frac{1}{4}\sin^2 2\theta \leq 1$

Thus $\frac{3}{4} \leq A \leq 1$

5. c. We have $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

and $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$\therefore$ L.H.S. $= \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$

$$= \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$
$$= \sin^2\frac{\pi}{8}\ \sin^2\frac{3\pi}{8}$$
$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$
$$= \frac{1}{4}\left[\left(1 - \cos\frac{\pi}{4}\right)\left(1 - \cos\frac{3\pi}{4}\right)\right]$$

$\left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}$$

6. c. The given expression is

$$\sqrt{3}\,\mathrm{cosec}\,20^\circ - \sec 20^\circ$$
$$= \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$
$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ\cos 20^\circ}$$
$$= 4\left[\frac{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ}{2\sin 20^\circ\cos 20^\circ}\right]$$
$$= 4\left[\frac{\sin 60^\circ\cos 20^\circ - \cos 60^\circ\sin 20^\circ}{\sin 2\times 20^\circ}\right]$$
$$= \frac{4\sin(60^\circ - 20^\circ)}{\sin 40^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4$$

7. b. $\sec 2x - \tan 2x = \dfrac{1-\sin 2x}{\cos 2x} = \dfrac{1-\cos 2\left(\dfrac{\pi}{4}-x\right)}{\sin 2\left(\dfrac{\pi}{4}-x\right)}$

$$= \frac{2\sin^2\left(\dfrac{\pi}{4}-x\right)}{2\sin\left(\dfrac{\pi}{4}-x\right)\cos\left(\dfrac{\pi}{4}-x\right)} = \tan\left(\frac{\pi}{4}-x\right)$$

8. d. $\sin\dfrac{\pi}{2n} + \cos\dfrac{\pi}{2n} = \dfrac{\sqrt{n}}{2}$

or $\quad \sin^2\dfrac{\pi}{2n} + \cos^2\dfrac{\pi}{2n} + 2\sin\dfrac{\pi}{2n}\cos\dfrac{\pi}{2n} = \dfrac{n}{4}$

$\Rightarrow \quad 1 + \sin\dfrac{\pi}{n} = \dfrac{n}{4} \quad$ or $\quad \sin\dfrac{\pi}{n} = \dfrac{n-4}{4}$

For $n = 2$, the given equation is not satisfied.

Considering that $n > 1$ and $n \neq 2$, $0 < \sin\dfrac{\pi}{n} < 1$

$\Rightarrow \quad 0 < \dfrac{n-4}{4} < 1 \quad \Rightarrow \quad 4 < n < 8$

9. c. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3(1 - \sin 2x)^2 + 6(1 + \sin 2x) + 4[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$

$= 3(1 - 2\sin 2x + \sin^2 2x) + (6 + 6\sin 2x) + 4\left[1 - \dfrac{3}{4}\sin^2 2x\right]$

$= 13 + 3\sin^2 2x - 3\sin^2 2x = 13$

10. b. Given, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

Now $\quad \sec^2\theta \geq 1 \Rightarrow \dfrac{4xy}{(x+y)^2} \geq 1$

or $\quad (x + y)^2 \leq 4xy$

or $\quad (x + y)^2 - 4xy \leq 0$

or $\quad (x - y)^2 \leq 0$

But for real values of x and y,

$(x - y)^2 \geq 0$ or $(x - y)^2 = 0$

$\therefore \quad x = y$

Also $\quad x + y \neq 0 \Rightarrow x \neq 0,\ y \neq 0$

11. c. $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$= (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta$

$= (4\sin\theta - 4\sin^3\theta)\sin\theta$

$= 4\sin^2\theta(1 - \sin^2\theta)$

$= 4\sin^2\theta\cos^2\theta$

$= (2\sin\theta\cos\theta)^2 = (\sin 2\theta)^2 \geq 0$

which is true for all θ.

12. c. $\alpha + \beta = \dfrac{\pi}{2} \quad$ or $\quad \alpha = \dfrac{\pi}{2} - \beta$

$\Rightarrow \quad \tan\alpha = \cot\beta \quad$ or $\quad \tan\alpha\tan\beta = 1 \qquad$ (i)

Again, $\quad \beta + \gamma = \alpha$ or $\gamma = \alpha - \beta$

$\Rightarrow \quad \tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{\tan\alpha - \tan\beta}{2}$ [using Eq. (i)]

or $\quad \tan\alpha = \tan\beta + 2\tan\gamma$

13. a. We are given that

$(\cot\alpha_1)(\cot\alpha_2)\cdots(\cot\alpha_n) = 1$

$\Rightarrow \quad (\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n)$
$= (\sin\alpha_1)(\sin\alpha_2)\cdots(\sin\alpha_n) \qquad$ (i)

Let $\quad y = (\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n)$ (to be maximum)

Squaring both sides, we get

$y^2 = (\cos^2\alpha_1)(\cos^2\alpha_2)\cdots(\cos_n^2)$

$= \cos\alpha_1\sin\alpha_1\cos\alpha_2\sin\alpha_2\cdots\cos\alpha_n\sin\alpha_n$ [using Eq. (i)]

$= \dfrac{1}{2^n}[\sin 2\alpha_1\sin 2\alpha_2\cdots\sin 2\alpha_n]$

As $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_n \leq \pi/2$, we have

$0 \leq 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_2 \leq \pi$

$\Rightarrow \quad 0 \leq \sin 2\alpha_1, \sin 2\alpha_2, \ldots, \sin 2\alpha_n \leq 1$

$\therefore \quad y^2 \leq \dfrac{1}{2^n}\times 1 \quad$ or $\quad y \leq \dfrac{1}{2^{n/2}}$

Therefore, the maximum value of y is $1/2^{n/2}$.

14. b. Given that $\sin\theta = 1/2$ and $\cos\phi = 1/3$, and θ and ϕ are acute angles. Therefore,

$\theta = \pi/6$ and $0 < \dfrac{1}{3} < \dfrac{1}{2}$

or $\quad \cos\pi/2 < \cos\phi < \cos\pi/3$ or $\pi/3 < \phi < \pi/2$

$\therefore \quad \dfrac{\pi}{3} + \dfrac{\pi}{6} < \theta + \phi < \dfrac{\pi}{2} + \dfrac{\pi}{6}$

or $\quad \dfrac{\pi}{2} < \theta + \phi < \dfrac{2\pi}{3} \quad \Rightarrow \quad \theta + \phi \in \left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$

15. b. $\theta \in \left(0, \dfrac{\pi}{4}\right) \Rightarrow \tan\theta < 1$ and $\cot\theta > 1$.

Let $\tan\theta = 1 - x$ and $\cot\theta = 1 + y$, where $x, y > 0$ and are very small, then

$t_1 = (1 - x)^{1-x}$, $t_2 = (1 - x)^{1+y}$, $t_3 = (1 + y)^{1-x}$, $t_4 = (1 + y)^{1+y}$

Clearly, $t_4 > t_3$ and $t_1 > t_2$. Also $t_3 > t_1$.

Thus, $\quad t_4 > t_3 > t_1 > t_2$.

16. d. In set P, $\sin\theta = (\sqrt{2} + 1)\cos\theta$

or $\quad \tan\theta = \sqrt{2} + 1$

In set Q, $(\sqrt{2} - 1)\sin\theta = \cos\theta$

or $\quad \tan\theta = \dfrac{1}{\sqrt{2}-1} = \sqrt{2} + 1 \quad \Rightarrow \quad P = Q$

Multiple Correct Answers Type

1. b. $3\left[\sin^4\left(\dfrac{3}{2}\pi - \alpha\right) + \sin^4(3\pi + \alpha)\right]$

$-2\left[\sin^6\left(\dfrac{1}{2}\pi + \alpha\right) + \sin^6(5\pi - \alpha)\right]$

$= 3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \sin^6\alpha)$

$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2[(\sin^2\alpha + \cos^2\alpha)^3$
$- 3\sin^2\alpha\cos^2\alpha(\sin^2\alpha + \cos^2\alpha)]$

$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2[1 - 3\sin^2\alpha\cos^2\alpha]$

$= 1$

2. b., c.

All are infinite geometric progression with common ratio < 1

$$x = \frac{1}{1-\cos^2\phi} = \frac{1}{\sin^2\phi}, \quad y = \frac{1}{1-\sin^2\phi} = \frac{1}{\cos^2\phi},$$

$$z = \frac{1}{1-\cos^2\phi\sin^2\phi}$$

Now, $xy + z = \dfrac{1}{\sin^2\phi\cos^2\phi} + \dfrac{1}{1-\sin^2\phi\cos^2\phi}$

$$= \frac{1}{\sin^2\phi\cos^2\phi(1-\sin^2\phi\cos^2\phi)}$$

or $\quad xy + z = xyz \qquad$ (i)

Clearly, $x + y = \dfrac{\sin^2\phi + \cos^2\phi}{\sin^2\phi\cos\phi} = xy$

$\therefore \quad x + y + z = xyz \qquad$ [using Eq. (i)]

3. c. For $\alpha = -\pi/2$, $\beta = -\pi/2$ and $\gamma = 2\pi$

$\sin\alpha + \sin\beta + \sin\gamma = -2$

Hence, the minimum value of the expression is negative.

4. c. We know that $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (irrational)

$\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (irrational)

$\sin 15° \cos 15° = \dfrac{1}{2}(2\sin 15°\cos 15°)$

$= \dfrac{1}{2}\sin 30° = \dfrac{1}{4}$ (rational)

$\sin 15° \cos 75° = \sin 15° \cos(90° - 15°)$

$= \sin 15° \sin 15° = \sin^2 15°$

$= \dfrac{1}{2}(1 + \cos 30°)$

$= \dfrac{1}{2}\left(1 - \dfrac{\sqrt{3}}{2}\right)$ (irrational)

5. a., b., c., d.

$$f_n(\theta) = \frac{\sin(\theta/2)}{\cos(\theta/2)} \times \left[\frac{2\cos^2(\theta/2)}{\cos\theta}\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right]$$

$$= \frac{\sin\theta}{\cos\theta}\left[\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right]$$

$$= \frac{\sin 2\theta}{\cos 2\theta}\left[\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right] = \tan 2^n\theta$$

$$\therefore \quad f_2\left(\frac{\pi}{16}\right) = \tan 4\frac{\pi}{16} = \tan\frac{\pi}{4} = 1$$

Similarly, $f_3(\pi/32)$, $f_4(\pi/64)$, and $f_5(\pi/128)$ are found to be $\tan\dfrac{\pi}{4} = 1$.

6. a., b.

$$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$$

$$3\sin^4 x + 2(1-\sin^2 x)^2 = \frac{6}{5}$$

$\Rightarrow \quad 25\sin^4 x - 20\sin^2 x + 4 = 0$

$\Rightarrow \quad \sin^2 x = \dfrac{2}{5} \therefore \cos^2 x = \dfrac{3}{5}$

$\therefore \quad \tan^2 x = \dfrac{2}{3}$

and $\quad \dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27} = \dfrac{1}{125}$

7. a., b.

For $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

Let $\quad \cos 4\theta = \dfrac{1}{3}$

$\Rightarrow \quad \cos 2\theta = \pm\sqrt{\dfrac{1+\cos 4\theta}{2}} = \pm\sqrt{\dfrac{2}{3}}$

$$f\left(\frac{1}{3}\right) = \frac{2}{2-\sec^2\theta} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$\Rightarrow f\left(\dfrac{1}{3}\right) = 1 - \sqrt{\dfrac{3}{2}}$ or $1 + \sqrt{\dfrac{3}{2}}$

Matching Column Type

1. b.

(q) $\cos x + \cos y + \cos z = 0$ and $\sin x + \sin y + \sin z = 0$

$\cos x + \cos y = -\cos z \qquad (1)$

and $\sin x + \sin y = -\sin z \qquad (2)$

Squaring and adding we get,

$1 + 1 + 2(\cos x\cos y + \sin x\sin y) = 1$

$\Rightarrow \quad 2 + 2\cos(x - y) = 1$

$\Rightarrow \quad 2\cos(x - y) = -1$

$\Rightarrow \quad \cos(x - y) = -\dfrac{1}{2}$

$\Rightarrow \quad 2\cos^2\left(\dfrac{x-y}{2}\right) - 1 = -\dfrac{1}{2}$

$\Rightarrow \quad \cos\left(\dfrac{x-y}{2}\right) = \dfrac{1}{2}$

(r) $\cos\left(\dfrac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x$

$= \cos x \sin 2x \sec x + \cos\left(\dfrac{\pi}{4} + x\right)\cos 2x$

$\Rightarrow \quad \left[\cos\left(\dfrac{\pi}{4} - x\right) - \cos\left(\dfrac{\pi}{4} + x\right)\right]\cos 2x$

$= (\cos x \sin 2x \sec x - \sin x \sin 2x \sec x)$

$$\Rightarrow \quad \frac{2}{\sqrt{2}} \sin x \cos 2x = (\cos x - \sin x) \sin 2x \sec x$$

$$\Rightarrow \quad \sqrt{2} \sin x \cos 2x = (\cos x - \sin x)\, 2 \sin x$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{1}{\cos x + \sin x}$$

$$\Rightarrow \quad x = \frac{\pi}{4} \quad \Rightarrow \quad \sec x = \sec \frac{\pi}{4} = \sqrt{2}$$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (7)

$$\frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

or $$\frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

or $$\frac{\left(2\sin\frac{\pi}{n}\cos\frac{2\pi}{n}\right)\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = 1$$

or $$\sin\frac{4\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \frac{4\pi}{n} + \frac{3\pi}{n} = \pi \Rightarrow n = 7.$$

2. (2) $$\frac{1}{4\cos^2\theta + 1 + \frac{3}{2}\sin 2\theta}$$

$$= \frac{1}{2[1+\cos 2\theta] + 1 + \frac{3}{2}\sin 2\theta}$$

$$= \frac{1}{2\cos 2\theta + \frac{3}{2}\sin 2\theta + 3}$$

Now

$$-\sqrt{2^2 + \left(\frac{3}{2}\right)^2} \le 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \sqrt{2^2 + \left(\frac{3}{2}\right)^2}$$

or $$-\frac{5}{2} \le 2\cos 2\theta + \frac{3}{2}\sin 2\theta \le \frac{5}{2}$$

$$\Rightarrow \quad \frac{1}{2} \le 2\cos 2\theta + \frac{3}{2}\sin 2\theta + 3 \le \frac{11}{2}$$

$$\Rightarrow \quad \frac{2}{11} \le \frac{1}{2\cos 2\theta + \frac{3}{2}\sin 2\theta + 3} \le 2$$

Hence, the maximum value is 2.

Fill in the Blanks Type

1. According to the given question, we have to express L.H.S. in the form

$$C_0 + C_1\cos x + C_2\cos 2x + \cdots + C_n\cos nx$$

Now, $\sin^3 x \sin 3x$

$$= \frac{3\sin x - \sin 3x}{4}\sin 3x$$

$$= \frac{3\sin x\sin 3x - \sin^2 3x}{4}$$

$$= \frac{3(\cos 2x - \cos 4x) - (1 - \cos 6x)}{8}$$

Hence, $n = 6$.

2. We know that A.M. ≥ G.M.

It implies that the minimum value of A.M. is obtained when A.M. = G.M.

Therefore, the quantities whose A.M. is being taken are equal. Thus,

$$\cos\left(\alpha + \frac{\pi}{2}\right) = \cos\left(\beta + \frac{\pi}{2}\right) = \cos\left(\gamma + \frac{\pi}{2}\right)$$

or $\sin\alpha = \sin\beta = \sin\gamma$

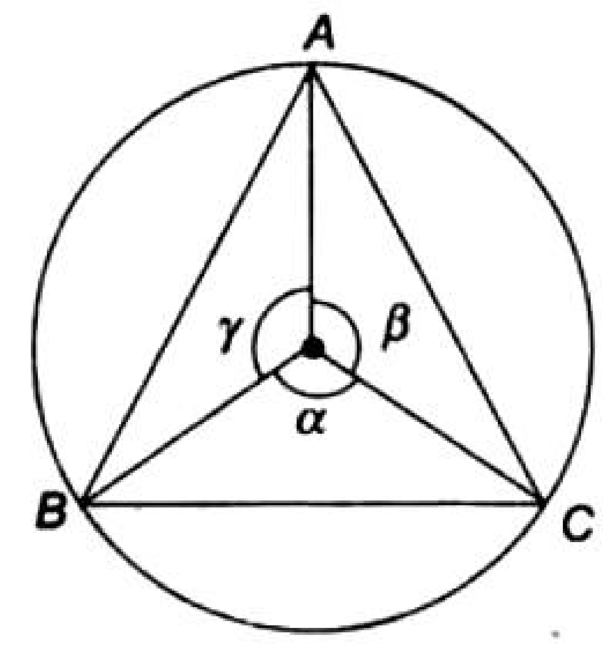
Also, $\alpha + \beta + \gamma = 360°$

or $\alpha = \beta = \gamma = 120° = \frac{2\pi}{3}$

∴ Minimum value of A.M.

$$= \frac{\cos\left(\frac{2\pi}{3} + \frac{\pi}{2}\right) + \cos\left(\frac{2\pi}{3} + \frac{\pi}{2}\right) + \cos\left(\frac{2\pi}{3} + \frac{\pi}{2}\right)}{3}$$

$$= \frac{-3\sin\frac{2\pi}{3}}{3} = -\frac{\sqrt{3}}{2}$$



3. $$\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$$

$$= \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{\pi}{2}\sin\left(\pi - \frac{5\pi}{14}\right)$$

$$\times \sin\left(\pi - \frac{3\pi}{14}\right)\sin\left(\pi - \frac{\pi}{14}\right)$$

$$= \sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}$$

$$= \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$$

$$= \left[\cos\left(\frac{\pi}{2} - \frac{\pi}{14}\right)\cos\left(\frac{\pi}{2} - \frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right)\right]^2$$

$$= \left[\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right]^2$$

$$= \left[\frac{1}{2\sin\pi/7}\left\{2\cos\frac{\pi}{7}\sin\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}\right]^2$$

$$= \left[\frac{1}{2^2 \sin \pi/7}\left\{2 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \cos\left(\pi - \frac{4\pi}{7}\right)\right\}\right]^2$$

$$= \left[\frac{1}{2^3 \sin \pi/7}\left(2 \sin \frac{4\pi}{7} \cos \frac{4\pi}{7}\right)\right]^2$$

$$= \left(\frac{1}{8 \sin \pi/7} \sin \frac{8\pi}{7}\right)^2$$

$$= \left(\frac{\sin(\pi + \pi/7)}{8 \sin \pi/7}\right)^2$$

$$= \left(\frac{-\sin \pi/7}{8 \sin \pi/7}\right)^2 = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$$

4. $k = \sin \frac{\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18}$

$$= \cos\left(\frac{\pi}{2} - \frac{\pi}{18}\right) \cos\left(\frac{\pi}{2} - \frac{5\pi}{18}\right) \cos\left(\frac{\pi}{2} - \frac{7\pi}{18}\right)$$

$$= \cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9}$$

$$= \frac{1}{2^3 \sin \frac{\pi}{9}} \sin \frac{8\pi}{9} = \frac{1}{8 \sin \frac{\pi}{9}} \sin \frac{\pi}{9} = \frac{1}{8}$$

$$\left[\because \sin \frac{8\pi}{9} = \sin\left(\pi - \frac{\pi}{9}\right) = \sin \frac{\pi}{9}\right]$$

Alternative solution:

$$k = \sin \frac{\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18}$$
$$= \sin 10° \sin 50° \sin 70°$$
$$= \sin 10° \sin(60° - 10°) \sin(60° - 10°)$$
$$= \frac{1}{4} \sin(3 \times 10°)$$
$$= \frac{\sin 30°}{4}$$
$$= \frac{1}{8}$$

5. $A + B = \pi/3 \quad \Rightarrow \quad \tan(A + B) = \sqrt{3}$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = \sqrt{3}$

or $\quad \frac{\tan A + \frac{y}{\tan A}}{1 - y} = \sqrt{3} \qquad$ [where $y = \tan A \tan B$]

or $\quad \tan^2 A + \sqrt{3}(y - 1) \tan A + y = 0$

For real value of $\tan A$,

$D \geq 0$

$\therefore \quad 3(y - 1)^2 - 4y \geq 0$

or $\quad 3y^2 - 10y + 3 \geq 0$

or $\quad (y - 3)\left(y - \frac{1}{3}\right) \geq 0$

or $\quad y \leq \frac{1}{3}$ or $y \geq 3$

But $A, B > 0$ and $A + B = \pi/3 \Rightarrow A, B < \pi/3$

$\Rightarrow \quad \tan A \tan B < 3$

Therefore, $y \leq 1/3$, i.e., the maximum value of y is 1/3.

6. We have $\frac{2}{\cos x} = \frac{1}{\cos(x - y)} + \frac{1}{\cos(x + y)}$

$$= \frac{2 \cos x \cos y}{\cos^2 x - \sin^2 y}$$

or $\quad \cos^2 x - \sin^2 y = \cos^2 x \cos y$

or $\quad \cos^2 x(1 - \cos y) = \sin^2 y$

or $\quad \cos^2 x \, 2 \sin^2 \frac{y}{2} = 4 \sin^2 \frac{y}{2} \cos^2 \frac{y}{2}$

or $\quad \cos^2 x = 2 \cos^2 \frac{y}{2}$

or $\quad \cos^2 x \sec^2 \frac{y}{2} = 2$

or $\quad \cos x \sec \frac{y}{2} = \pm\sqrt{2}$

True/False Type

1. True. $\tan A = \frac{1 - \cos B}{\sin B} = \frac{2 \sin^2 \frac{B}{2}}{2 \sin \frac{B}{2} \cos \frac{B}{2}} = \tan \frac{B}{2}$

Hence, $\quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2 \tan \frac{B}{2}}{1 - \tan^2 \frac{B}{2}} = \tan B$

Therefore, the statement is true.

Subjective Type

1. We have $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

$$= \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1} \times \frac{1}{2m+1}}$$

$$= \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1$$

$\Rightarrow \quad \alpha + \beta = n\pi + \pi/4$, where $n \in Z$.

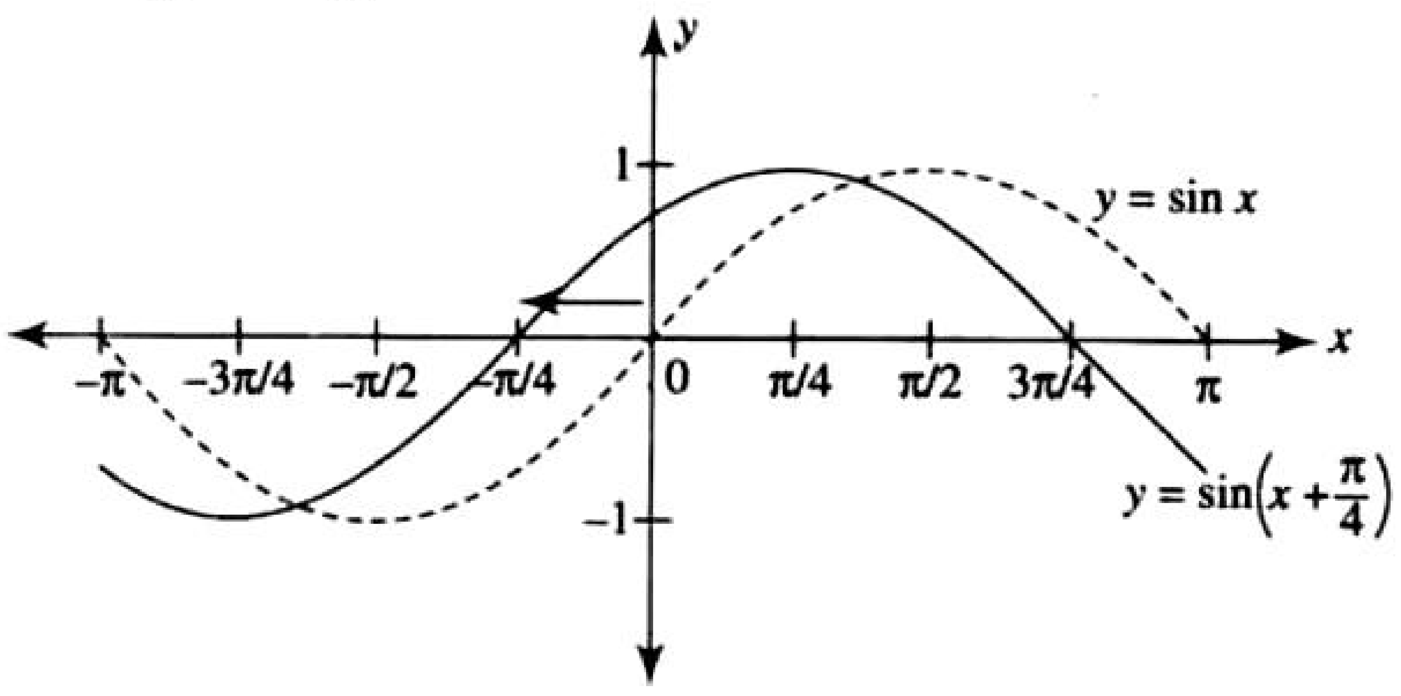
2. a. To draw the graph of $y = \frac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

$$y = \frac{1}{\sqrt{2}}(\sin x + \cos x) = \sin\left(x + \frac{\pi}{4}\right)$$

To draw the graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ we first draw the graph of $y = \sin x$ and then shift the graph $\frac{\pi}{4}$ to the left along x-axis.

Also for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, we have $-\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$

Thus graph of $y = \sin\left(x+\frac{\pi}{4}\right)$ is to be considered for $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$



b. We have

$$\cos(\alpha+\beta) = \frac{4}{5} \text{ and } \sin(\alpha-\beta) = \frac{5}{13}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{3}{4} \text{ and } \tan(\alpha-\beta) = \frac{5}{12}$$

$$\tan 2\alpha = \tan[(\alpha+\beta)+(\alpha-\beta)]$$

$$= \frac{\tan(\alpha-\beta)+\tan(\alpha+\beta)}{1-\tan(\alpha-\beta)\tan(\alpha+\beta)}$$

$$= \frac{\frac{5}{12}+\frac{3}{4}}{1-\left(\frac{5}{12}\right)\left(\frac{3}{4}\right)} = \frac{56}{33}$$

3. We have

$$5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3$$

$$= 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3}+3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

Now, $-\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

$$\le \sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$$

$$\Rightarrow \quad -4 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 10$$

4. Given $\alpha+\beta-\gamma=\pi$ and we want to prove that

$\sin^2\alpha+\sin^2\beta-\sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$

L.H.S. $= \sin^2\alpha+\sin^2\beta-\sin^2\gamma$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(\beta-\gamma)$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(\pi-\alpha)$ $\quad (\because \alpha+\beta-\gamma=\pi)$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin\alpha$

$= \sin\alpha\,[\sin\alpha + \sin(\beta+\gamma)]$

$= \sin\alpha[\sin[\pi-(\beta-\gamma)] + \sin(\beta+\gamma)]$

$= \sin\alpha[\sin(\beta-\gamma) + \sin(\beta+\gamma)]$

$= \sin\alpha\,[2\sin\beta\cos\gamma]$

$= 2\sin\alpha\sin\beta\cos\gamma$

= R.H.S.

5. We have,

$$\cos\theta+\sin\theta = \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right]$$

$$= \sqrt{2}\sin(\pi/4+\theta)$$

$\therefore \quad \cos\theta+\sin\theta \le \sqrt{2} < \pi/2$

$\therefore \quad \cos\theta+\sin\theta < \pi/2$

or $\quad \cos\theta < \pi/2 - \sin\theta$ $\quad$ (i)

As $\theta \in [0, \pi/2]$ in which $\sin\theta$ increases. Taking sin on both the sides of Eq. (i), we get

$\sin(\cos\theta) < \sin(\pi/2 - \sin\theta)$

or $\quad \sin(\cos\theta) < \cos(\sin\theta)$

or $\quad \cos(\sin\theta) > \sin(\cos\theta)$ $\quad$ (ii)

6. L.H.S. $= \sin 12° \sin 48° \sin 54°$

$$= \frac{1}{2}[2\sin 12°\cos 42°]\sin 54°$$

$$= \frac{1}{2}[\sin 54° - \sin 30°]\sin 54°$$

$$= \frac{1}{2}\left[\sin^2 54° - \frac{1}{2}\sin 54°\right]$$

$$= \frac{1}{4}\left[2\sin^2 54° - \sin 54°\right]$$

$$= \frac{1}{4}\left[2\left(\frac{1+\sqrt{5}}{4}\right)^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{4}\left[2\left(\frac{1+5+2\sqrt{5}}{16}\right) - \left(\frac{1+\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{4}\times\frac{1}{8}[6+2\sqrt{5}-2-2\sqrt{5}]$$

$$= \frac{1}{32}\times 4 = \frac{1}{8} = \text{R.H.S.}$$

7. We know that

$$\cos A\cos 2A\cos 4A \cdots \cos 2^n A$$

$$= \frac{1}{2^{n+1}\sin A}\sin\left(2^{n+1}A\right)$$

$$\therefore \quad 16\cos\frac{2\pi}{15}\cos 2\left(\frac{2\pi}{15}\right)\cos 2^2\left(\frac{2\pi}{15}\right) \times \cos 2^3\left(\frac{2\pi}{15}\right)$$

$$= 16\frac{\sin\left(2^4 A\right)}{2^4 \sin A} \quad \text{(where } A = 2\pi/15\text{)}$$

$$= 16\frac{\sin(32\pi/15)}{16\sin 2\pi/15} = \frac{\sin(32\pi/15)}{\sin(2\pi+2\pi/15)}$$

$$= \frac{\sin(32\pi/15)}{\sin(32\pi/15)} = 1$$

8. We know that

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

or $\quad \dfrac{1-\tan^2\alpha}{\tan\alpha} = 2\cot 2\alpha$

or $\cot\alpha - \tan\alpha = 2\cot 2\alpha$ (i)

Now we have to prove

$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$

L.H.S. $= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(2\cot 8\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(\cot 4\alpha - \tan 4\alpha)$ [using Eq. (i)]

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4\cot 4\alpha - 4\tan 4\alpha$

$= \tan\alpha + 2\tan 2\alpha + 2(2\cot 4\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 2(\cot 2\alpha - \tan 2\alpha)$ [using Eq. (i)]

$= \tan\alpha + 2\cot 2\alpha$

$= \tan\alpha + (\cot\alpha - \tan\alpha)$ [using Eq. (i)]

$= \cot\alpha =$ R.H.S.

9. Given that in ΔABC, A, B, and C are in A.P. Therefore,

$$A + B = 2B$$

Also $A + B + C = 180°$

$\Rightarrow$ $B + 2B = 180°$ or $B = 60°$

Also given that

$$\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = \frac{1}{2}$$

$\Rightarrow$ $\sin(2A + 60°) = \sin(C - A) = -\sin(60° + 2C)$

$$= \frac{1}{2} \quad \text{(i)}$$

From Eq. (i), we have

$$\sin(2A + 60°) = \frac{1}{2}$$

$\Rightarrow$ $2A + 60° = 150°$

or $2A = 90°$

or $A = 45°$

$\Rightarrow$ $C = \pi - A - B = 75°$

10. Let $y = \dfrac{\tan x}{\tan 3x}$

$$= \frac{\tan x\left(1 - 3\tan^2 x\right)}{3\tan x - \tan^3 x}$$

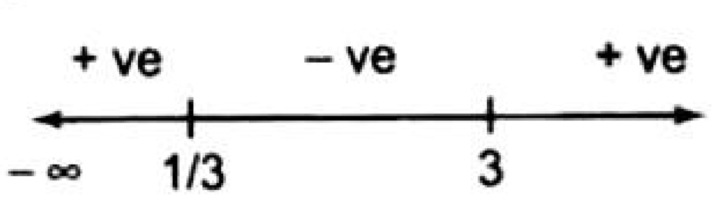
$$= \frac{1 - 3\tan^2 x}{3 - \tan^2 x}$$

or $3y - (\tan^2 x)y = 1 - 3\tan^2 x$

or $(y - 3)\tan^2 x = 3y - 1$

or $\tan^2 x = \dfrac{3y - 1}{y - 3}$

or $\dfrac{3y - 1}{y - 3} \geq 0$ (L.H.S. is a prefect square)



From the sign scheme $y < \dfrac{1}{3}$ or $y \geq 3$

Thus, y never lies between 1/3 and 3.

11. $S = \displaystyle\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n}$

$$= (n-1)\cos\frac{2\pi}{n} + (n-2)\cos 2\frac{2\pi}{n} + \cdots + 1\cos(n-1)\frac{2\pi}{n} \quad \text{(i)}$$

We know that $\cos\theta = \cos(2\pi - \theta)$. Replacing each angle θ by $2\pi - \theta$ in Eq. (i), we get

$$S = (n-1)\cos(n-1)\frac{2\pi}{n} + (n-2)\cos(n-2)\frac{2\pi}{n} + \cdots + 1\cos\frac{2\pi}{n} \quad \text{[using Eq. (i)]} \quad \text{(ii)}$$

Adding terms having the same angle and taking n common, we have

$$2S = n\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \cdots + \cos(n-1)\frac{2\pi}{n}\right]$$

$$= n\left[\frac{\sin(n-1)\frac{\pi}{n}}{\sin\frac{\pi}{n}}\cos\frac{\frac{2\pi}{n} + (n-1)\frac{2\pi}{n}}{2}\right]$$

$$= n\left[\frac{\sin\left(\pi - \frac{\pi}{n}\right)}{\sin\frac{\pi}{n}}\cdot\cos\pi\right] = -n$$

$\therefore S = -n/2$

12. Given that

$$2\sin t = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}, t \in [-\pi/2, \pi/2]$$

This can be written as

$$(6\sin t - 5)x^2 + 2(1 - 2\sin t)x - (1 + 2\sin t) = 0$$

For the given equation to hold, x should be a real number; therefore, the above equation should have real roots, i.e., $D \geq 0$. Thus,

$$4(1 - 2\sin t)^2 + 4(6\sin t - 5)(1 + 2\sin t) \geq 0$$

or $16\sin^2 t - 8\sin t - 4 \geq 0$

or $(4\sin^2 t - 2\sin t - 1) \geq 0$

or $4\left(\sin t - \dfrac{\sqrt{5}+1}{4}\right)\left(\sin t + \dfrac{\sqrt{5}-1}{4}\right) \geq 0$

or $\sin t \leq -\left(\dfrac{\sqrt{5}-1}{4}\right)$ or $\sin t \geq \dfrac{\sqrt{5}+1}{4}$

or $\sin t \leq \sin(-\pi/10)$ or $\sin t \geq \sin(3\pi/10)$

or $t \leq -\pi/10$ or $t \geq 3\pi/10$

(Note that $\sin x$ is an increasing function from $-\pi/2$ to $\pi/2$.)

Therefore, the range of t is $[-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2]$.